Exponential Functions

Lets start with an example:

Say you have \$ P_0 today in bank & you get 4% annual growth.

End of Year 1 $\rightsquigarrow \$ (P_0 + P_0 \cdot (0.04)) = P_0 (1.04)$

$2 \rightsquigarrow \$ (P_0 (1.04) + P_0 (1.04)(0.04))$

$= P_0 (1.04)(1 + 0.04)$

$= P_0 (1.04)(1.04)$

$= P_0 (1.04)^2$

$3 \rightsquigarrow \$ P_0 (1.04)^3$

$\vdots$

$r \rightsquigarrow \$ P_0 (1.04)^r$

More generally we get a special type of function

$f(x) = b^x$, where $b > 0$, $b \neq 1$

Exponential function with base b & exponent x.
domain $(-\infty, \infty)$ & range $(0, \infty)$

Note: It is different from x^n, $n \geq 1$. Both grows, but exponential functions grows faster than power functions.

Most Common Applications: ① Growth of money / Compound Interest
② Growth of Bacterias / Cells.

Rules: ① $b^x \cdot b^y = b^{x+y}$

② $\frac{b^x}{b^y} = b^{x-y}$, $b^{-y} = \frac{1}{b^y}$

③ $(b^x)^y = b^{xy}$

④ $(a \cdot b)^x = a^x \cdot b^x$,

⑤ $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$.

Simplyfy : $\dfrac{(x^3y^{-1})^2}{(xy^2)^{-2}}$

A special Number 'e'

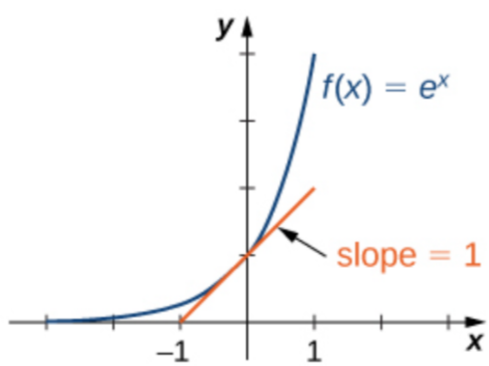
Observation :

m	10	$100 = 10^2$	$1000 = 10^3$	$10{,}000 = 10^4$	$100{,}000 = 10^5$	10^6
$\left(1+\frac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.71828	2.718280

If we take $m \to \infty$, then $\left(1+\frac{1}{m}\right)^m \to$ some fixed value $= e$

$$e \approx 2.718282$$

$\boxed{f(x) = e^x} \longrightarrow$ Natural Exponential function.



Logarithmic Functions

Exponential functions are One-to-One

$\Downarrow$

Exponential functions possesses inverse. Called Logarithmic Functions.

So, Domain of Log. Function is $(0, \infty)$

Range of Log. Function is $(-\infty, \infty)$

$$\log_b(x) = y \quad \text{if \& only if} \quad b^y = x.$$

When we use the base 'e', ie, $\log_e$ function, we call it natural logarithm. (ln)

Prop: ① $\log_b(b^x) = x$ & $b^{\log_b(y)} = y$.

② $\log_e(e^n) = \ln(e^n) = n$.

③ $\log_b(1) = 0$, for any base b

④ $\log_b(ac) = \log_b(a) + \log_b(c)$ ⇝ product rule

⑤ $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$ ⇝ Quotient rule.

⑥ $\log_b(a^r) = r \log_b(a)$ ⇝ power rule.

⑦ $a^x = b^{x \log_b a}$, for any real x.

⑧ $\log_a x = \dfrac{\log_b x}{\log_b a} = \dfrac{\ln x}{\ln a}$

→ Base change rules.

Eg. $\ln(2x) - \ln(x^6) = 0$

$\Rightarrow \ln\left(\frac{2x}{x^6}\right) = 0$, quotient rule

$\Rightarrow \ln\left(\frac{2}{x^5}\right) = 0$

$\Rightarrow \frac{2}{x^5} = 1 \Rightarrow x^5 = 2 \Rightarrow x = \sqrt[5]{2}$